Diagonalization of Linear Operator

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Announcement

- 明天 (5/5, 週四) 助教會來講解作業二
- 會有 bonus 的作業

Review

$$P = [p_1 \quad \cdots \quad p_n]$$

eigenvector

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

eigenvalue

An n x n matrix A is diagonalizable ($A = PDP^{-1}$)

=

The eigenvectors of A can form a basis for R^n.

$$det(A - tI_n)$$
$$= (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2} \dots (t-\lambda_k)^{m_k}(\dots\dots)$$

Eigenvalue: λ_1 λ_2 …… λ_k

Eigenspace: ……

$\{$ Basis for λ_1 Basis for λ_2 Basis for λ_3 $\}$

Independent Eigenvectors

Diagonalization of Linear Operator

- Example 1: $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 8x_1 + 9x_2 \\ -6x_1 - 7x_2 \\ 3x_1 + 3x_2 - x_3 \end{bmatrix}$

det

The standard matrix is $A = \begin{bmatrix} 8 \boxed{-t} & 9 & 0 \\ -6 & -7 \boxed{-t} & 0 \\ 3 & 3 & -1 \boxed{-t} \end{bmatrix}$

$\Rightarrow$ the characteristic polynomial is $-(t+1)^2(t-2)$

Eigenvalue -1:

$\mathcal{B}_1 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Eigenvalue 2:

$\mathcal{B}_2 = \left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \right\}$

$\Rightarrow \mathcal{B}_1 \cup \mathcal{B}_2$ is a basis of $\mathcal{R}^3$

Diagonalization of Linear Operator

- Example 2: $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + x_2 + 2x_3 \\ x_1 - x_2 \\ 0 \end{bmatrix}$

det -t -t -t

The standard matrix is $A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

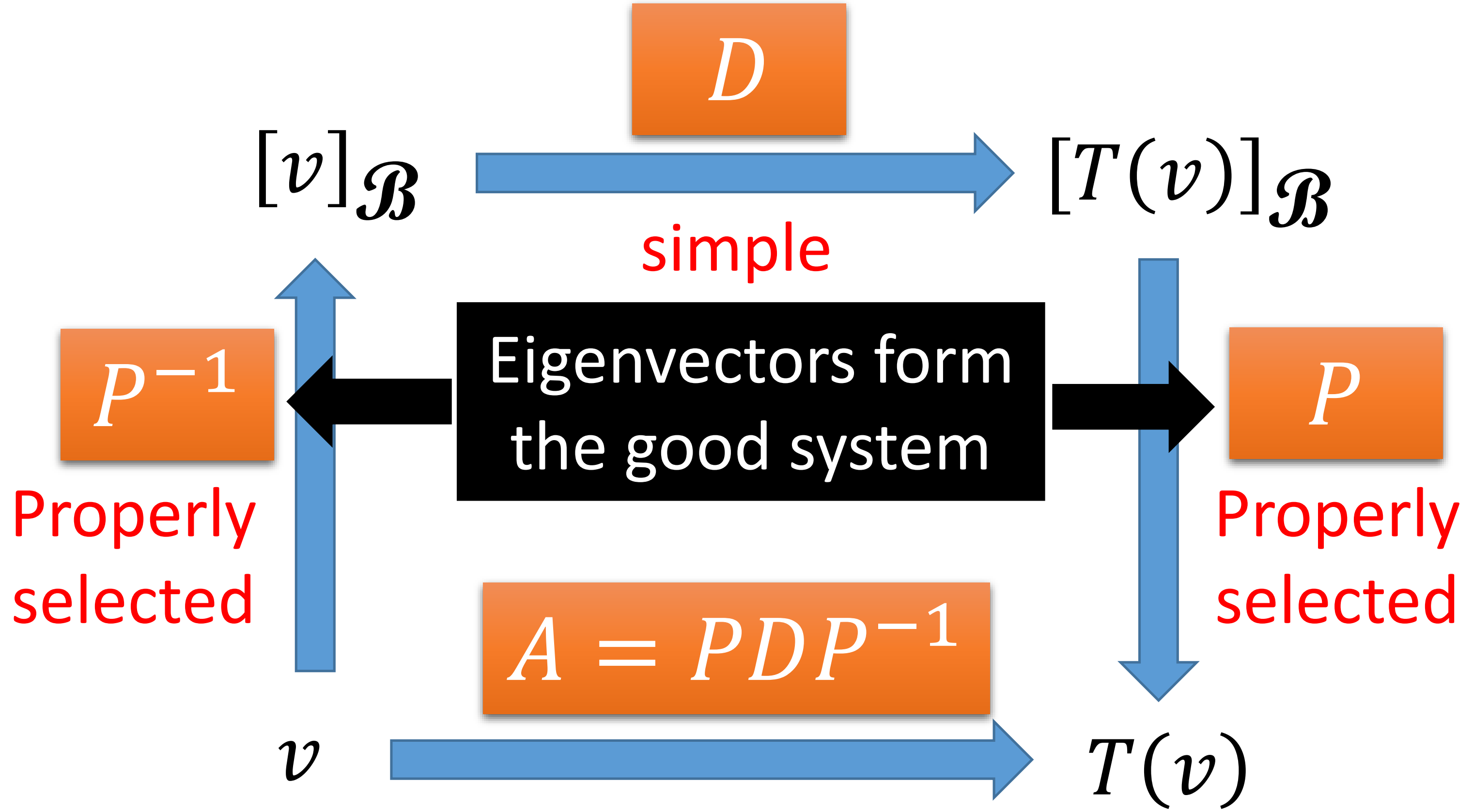
This lecture

- Reference: Chapter 5.4

$$
\begin{tikzcd}
{[v]_{\mathcal{B}}} \arrow[r, "{[T]_{\mathcal{B}}}", "\text{simple}"'] & {[T(v)]_{\mathcal{B}}} \arrow[d, "B", "\text{Properly selected}"'] \\
v \arrow[u, "B^{-1}", "\text{Properly selected}"'] \arrow[r, "A"] & T(v)
\end{tikzcd}
$$

Diagonalization of Linear Operator

- If a linear operator T is diagonalizable



Diagonalization of Linear Operator

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 8x_1 + 9x_2 \\ -6x_1 - 7x_2 \\ 3x_1 + 3x_2 - x_3 \end{bmatrix}$$

-1: $\mathcal{B}_1 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

2: $\mathcal{B}_2 = \left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \right\}$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
{[v]_{\mathcal{B}}} \arrow[r, "{[T]_{\mathcal{B}}}", "{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}"'] & {[T(v)]_{\mathcal{B}}} \arrow[d, "{\begin{bmatrix} -1 & 0 & 3 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}}"] \\
v \arrow[u, "{\begin{bmatrix} -1 & 0 & 3 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}^{-1}}"] \arrow[r, "{\begin{bmatrix} 8 & 9 & 0 \\ -6 & -7 & 0 \\ 3 & 3 & -1 \end{bmatrix}}"] & T(v)
\end{tikzcd}$$

Another Application of Eigenvector: Page Rank

- Reference:
  - THE $25,000,000,000 EIGENVECTOR: THE LINEAR ALGEBRA BEHIND GOOGLE
    - http://userpages.umbc.edu/~kogan/teaching/m430/GooglePageRank.pdf
  - A SURVEY OF EIGENVECTOR METHODS FOR WEB INFORMATION RETRIEVAL
    - http://doradca.oeiizk.waw.pl/survey.pdf